Real Analysis

- According to *Principles of Mathematical Analysis* by Walter Rudin (Chapter 1-4)

1 The Real and Complex Number

Set: a collection of objects.

Proper subset: if $A \subsetneqq B$, then call A a proper subset of B.

Set equality: if $A \subset B$ and $B \subset A$, then $A = B$; otherwise $A \neq B$.

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Complement: $A^{\mathsf{c}} = \{x : x \notin A\}$.

Set Minus: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Set Product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. (a, b) is not a set, but a ordered pair.

Set Relationship

Binary relationship: A binary relationship R is a subset of $A \times B$. Write $(a, b) \in R$ as aRb.

Function: a function from A to B is a relation such that if aFb and bFa, then $a = b$, write as $F(a) = b$.

Equivalence: an equivalence relation R on a set $\mathbb{S}$ is a relation on $\mathbb{S} \times \mathbb{S}$ such that:

1. Reflexive property: aRa is true,
2. Symmetric property: $aRb \Rightarrow bRa$,
3. Transitive property: if aRb and bRc, then aRc.

Equivalence class

Inequalities

Order Set: an order on $\mathbb{S}$ is a relation $<$ satisfying:

1. Trichotomy: if $x, y \in \mathbb{S}$, then only one of these is true: $x < y, x = y$, or $y < x$.
2. Transitivity: if $x, y, z \in \mathbb{S}$, and if $x < y$ and $y < z$, then $x < z$.

Rational Number: $\mathbb{Q} = \{\frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0\}$

Cancellation law in $\mathbb{Z}$: if $ab = ac$, and $a \neq 0$, then $b = c$.

Addition: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ (why not $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$?)

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Proof: $x^2 = 2$, no solution in $\mathbb{Q}$.

Fields: a set with two operations: $+$ and $\times$

Axioms of addition:

A1: Closed under addition: if $x \in F$ and $y \in F$, then $x + y \in F$.

A2: Commutative: $x + y = y + x$.

A3: Associative: $x + (y + z) = (x + y) + z$.

A4: Additive identity element '0': for all $x \in F$, $x + 0 = x$.

A5: Additive inverse: for every $x \in F$, there is a corresponding "additive inverse" $y \in F$ such that $x + y = 0$, which represented as "$-x$".

Axioms of multiplication:

M1: Closed under multiplication: if $x \in F$ and $y \in F$, then $x \times y \in F$.

M2: Commutative: $x \times y = y \times x$.

M3: Associative: $x \times (y \times z) = (x \times y) \times z$.

M4: Multiplicative identity: F contains a "multiplicative identity" element $1 \neq 0$, such that for all $x \in F$, $x \times 1 = x$.

M5: Multiplicative inverse: For every $x \in F$, if $x \neq 0$, there is a corresponding "multiplicative inverse" $y \in F$ such that $x \times y = 1$, which represented as "$\frac{1}{x}$".

$\mathbb{Q}$ is a field: 0 element is $\frac{0}{1}$, 1 element is $\frac{1}{1}$.

$\mathbb{Z}$ is not a field (fails M5).

Order feild

$\mathbb{Q}$ is an order feild.

Boundary

Upper bound: an upper bound of a set $\mathbb{S}$ is an element of k which is greater than or equal to every element of $\mathbb{S}$.

A set which has an upper bound is said to be bounded above.

Least Upper Bound (LUB): the greatest element in upper bounds, also called "supermun" (SUP).

LUB Property: a set $\mathbb{S}$ has the LUB if every non-empty subset of $\mathbb{S}$, which has an upper bound, also has a LUB $\in \mathbb{S}$.

1.γ is a UB of $A \Leftrightarrow supA \leq \gamma$.

2.$\forall x \in A, a \leq \gamma \Leftrightarrow supA \leq \gamma$.

3.$\forall x \in A, a < \gamma \Leftrightarrow supA \leq \gamma$.

4.$\gamma < supA \Rightarrow \exists a \in A$, such that $\gamma < a \leq supA$.

5. if$A \in B \Rightarrow supA \leq supB$.

6. if$supA \leq supB$ and $supA \geq supB \Rightarrow supA = supB$.

Lower bound: lower bound works the opposite way of upper bound, and so does the Greatest Lower Bound (GLB), as LUB.

Real number

Dedekind cut: a partitioning of $\mathbb{Q}$ into two non-empty sets, which is represented as α:

1. not trivial: not $\emptyset$, and not all of $\mathbb{Q}$.

2. closed downwards (left): if any $x \in \mathbb{Q}$ is in the cut α, all rationals $y < x$ are also in the cut. $\forall y, x \in \mathbb{Q}, y < x (x \in \alpha \Rightarrow y \in \alpha)$.

3. no largest element: for any $x \in \alpha$, there exists a $y \in \alpha$ such that $y > x$.

Real number: define real numbers as the set of all Dedekind cuts, represented as $\mathbb{R}$.

Order on Dedekind cut $\alpha < \beta$: $\alpha \subset \beta$ but $\alpha \neq \beta$.

Addition on Dedekind cut: $\alpha + \beta = \{r + s : r \in \alpha \text{ and } s \in \beta\}$.

How to proof $\alpha + \beta$ is a cut?

Multiplication on Dedekind cut: if $\alpha, \beta \in \mathbb{R}_+$, then $\alpha\beta = \{p : p < rs, \text{ for } r \in \alpha, s \in \beta, r, s > 0\}$

Multiplication identity: $1^* = q < 1 : q \in \mathbb{Q}$.

$\mathbb{Q}$ is a subfeild in $\mathbb{R}$.

$\mathbb{R}$ is an ordered field, which estends $\mathbb{Q}$ and has the LUB property.

Archimedean properity: if $x, y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$, such that $nx < y$.

$\mathbb{Q}$ is dense in $\mathbb{R}$: between any $x, y \in \mathbb{R}, x < y, \exists q \in \mathbb{Q}$ such that $x < q < y$.

Complex Numbers: $\mathbb{C} = \mathbb{R} \cup \{-\infty, +\infty\}$

$\mathbb{C}$ are the field of numbers $x + iy$, where x and y are real numbers and i is the imaginary unit.

We write $\mathbb{C}$ in the form of (x, y), which means $x + iy$.

$i = \sqrt{-1}$, then $i^2 = 1$.

Operations: $'+' \Rightarrow (a, b) + (c, d) = (a + b, c + d)$.

$'\times' \Rightarrow (a, b) \times (c, d) = (ac - bd, ad + bc)$.

$\mathbb{C}$ is not a field.

Conjugate: if a and b are reals, and $z = a + bi$, then the complex number $\bar{z} = a - bi$.

Write $a = Re(z)$, and $b = Im(z)$.

$|\bar{z}| = |z|$

$|zw| = |z||w|$

$Re(z) < |z| + |w|$

Euclidean Spaces: a real vector space which is equipped with a fixed symmetric bilinear form $\zeta : \mathbf{E} \times \mathbf{E} \to \mathbf{R}$, which is also positive definite ($\zeta(x, x) > 0$ for every $x \neq 0$)

The standard example of a Euclidean space is $\mathbf{R}^n$, it is defined under the inner product that $(x_1, \ldots, x_n) \times (y_1, \ldots, y_n) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

Axioms: the real number $\zeta(x, y)$ is also called the inner product (or scalar product) of x and y.

$$\zeta(x_1 + x_2, y) = \zeta(x_1, y) + \zeta(x_2, y)$$

$$\zeta(x, y_1 + y_2) = \zeta(x, y_1) + \zeta(x, y_2)$$

$$\zeta(\lambda x, y) = \lambda \zeta(x, y)$$

$$\zeta(x, \lambda y) = \lambda \zeta(x, y)$$

$$\zeta(x, y) = \lambda \zeta(y, x)$$

Inner product: $\vec{x} \cdot \vec{y} = \sum x_i y_i$.

Norm (Length): $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})}$.

Cauchy-Schwarz inequality: if $a_1, \ldots, a_n; b_1, \ldots, b_n \in \mathbf{C}$, then $|\sum_{i=1}^{k} a_i \bar{b}_i|^2 \leq \sum_{i=1}^{k} |a_i|^2 \cdot \sum_{i=1}^{k} |b_i|^2$. (how to proof?)

2 Basic Topology

Natural numbers: set of numbers, denoted by $\mathbb{N}$, starting at 0, such that every number has a unique successor, and every number other than zero has a unique predecessor.

Addition: $n + S(m) = S(n + m), \forall n, m \in \mathbb{N}$, where $S(m)$ is the successor function.

Multiplication: $n \times 0 = 0, \forall n \in \mathbb{N}$

$$n \times S(m) = n \times m + n, \forall n, m \in \mathbb{N}.$$

Identity elements: additive identity element is 0, and multiplicative identity element is 1.

Induction

Priciple of Induction: let $\mathbb{S}$ be a subset of $\mathbb{N}$, $\mathbb{S} = \mathbb{N}$ if it satisfies:

1. $1 \in \mathbb{S}$.
2. if $k \in \mathbb{S} \Rightarrow (k + 1) \in \mathbb{S}$.

Methods of induction proof: let $P(n)$ be some statement that is indexed by $n \in \mathbb{N}$.

(a) the base case: proof P(1) is true. (not necessary for n=1);

(b) inductive step: assume P(k) is true,

proof P(k+1) is also true.

$\Rightarrow$ by the principle of induction, $P(n)$ is true for all n.

Set Relationship

Function $f : A \to B$, call A domain, B codomain.

Injection: f is $1 - 1$ injection when $f(x) = f(y) \Rightarrow x = y$.

Surjection: f is onto (a surjection) when $f(A) = B$.

Bijection $A \sim B$: f is a bijection if it is both 1-1 and onto.

Finite Sets

Ordinal numbers are used to identify the position of an entry in an ordered list. According to Von Neumann's definition, an ordinal is the the well-ordered set of all smaller ordinals:
$J_n = 0, 1, \ldots, n - 1$.

Finite Sets: any set that can be assigned a one-to-one correspondence, or isomorphism with any ordinal ($A \sim J_n$).

Otherwise, call A infinite.

$\mathbb{N}$ is infinite. (how to prove?)

A set is infinite if and only if it can be put into a 1-1 with itself.

Countable Set: any set that is isomorphic with $\mathbb{N}$

$\mathbb{N}$ is countable.

$\mathbb{Z}$ is countable.

$\mathbb{Q}$ is countable. (how to proof?)

Every infinite subset $\mathbb{E}$ of countable set $\mathbb{A}$ is countable.

$\mathbb{R}$ is uncountable.

Powerset: the set of all subsets of A (2^A).

Cantor's theorem: for any set A, $A \nsim 2^A$.

The countable union of countable sets is countable.

Metric Space

A set $\mathbb{X}$ is a metric space, if any two point p and q of $\mathbb{X}$ can be assigned to a function $d(p, q) : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ (called a "metric" or, a "distance function"), satisfying the properties:

1. Non-negative: $d(p, q) \geq 0$ if $p \neq q$, and $d(p, q) = 0$ iff $p = q$.
2. Symmetric: $d(p, q) = d(q, p)$.
3. Triangle Inequality: $d(p, r) + d(r, q) \geq d(p, q)$, for any $r \in \mathbb{X}$.

Metric space is fully represented by as $(\mathbb{X}, d)$, in which $\mathbb{X}$ is a set and d is a metric.

Limit Point

An open ball $N_r(x)$ is a neighborhood f $\mathbb{X}$ defined as $\{y|d(x, y) < r\}, r > 0\},$.

The open ball for $n = 1$ is called an open interval.

A closed ball $\bar{N_r}(x)$ is a neighborhood f $\mathbb{X}$ defined as $\{y|d(x, y) \leq r\}, r > 0\}$, which includes the boundary.

Limit points: $p, p \in \mathbb{X}$ is a limit point of $\mathbb{E}, |\mathbb{E}| \leq |\mathbb{X}|$ if every neighborhood of p includes at least one element $q \neq p$ such that $q \in \mathbb{E}$.

Interior point: A point $p, p \in \mathbb{R}$ is an interior point of set $\mathbb{S}$ if it has a neighborhood $\mathbb{X}$ is entirely inside $\mathbb{S}$.

Isolated point: p is an isolated point of $\mathbb{E}$ is $p \in \mathbb{E}$, but p is not a limit point of $\mathbb{E}$.

If p is a limit point of $\mathbb{E}$, every neighborhood of p contains infinitely many points of $\mathbb{E}$. (how to prove it?)

Open sets: a set $\mathbb{E}$ is open if every point in $\mathbb{E}$ is an interior point of $\mathbb{E}$.

Closed set is a set that includes all its limit points.

Closed set is not the opposite of open set.

Closure of a set $\mathbb{E}$ is the set $\bar{\mathbb{E}}$ includes $\mathbb{E}$ and $\mathbb{E}'$ which contains all limit points of $\mathbb{E}$.

$\bar{\mathbb{E}}$ is a closed set. (how to proof it?)

$\mathbb{E}$ is closed if and only if $\bar{\mathbb{E}}$.

Complement of open sets are closed.

Complement of closed sets are open.

Arbitary union of open sets is open.

Arbitary interestion of closed sets is closed.

Finite interesction of open sets is open.

Finite union of closed sets is closed.

Dense set: $\mathbb{E}$ is dense in metric $\mathbb{X}$ if every point of $\mathbb{X}$ is a limit point of $\mathbb{E}$ or in $\mathbb{E}$, $\bar{\mathbb{E}} = \mathbb{X}$ or every open set of $\mathbb{X}$ contains a point of $\mathbb{E}$.

Compact Set

Open cover: An open cover of $\mathbb{E}$ in $\mathbb{X}$ is a collection of sets S_α whose union covers $\mathbb{E}$.

Subcover: a subcover of S_α is a subcollection S_{α_i} that still covers $\mathbb{E}$.

Compact set: a set is compact if every open cover contains a finite subcover.

Finite sets are compact.

Bounded set: a set $\mathbb{K}$ is bounded if there exist some balls that contains the entire set.

Relative compactness: $Y \subset X, K$ compact in Y if and only if K compact in X.

A set is compact if and only if it is bounded and closed.

A closed subawt B of a compact set K is also compact.

k-cells are compact in $\mathbb{R}^k$.

Heine-Borel Theorem: in $\mathbb{R}^n$, $\mathbb{K}$ is compact iff $\mathbb{K}$ is closed and bounded.

K is compact if and only if every infinite subset $\mathbb{K}$ has a limit point in $\mathbb{K}$.

Connect Sets

The Cantor set T_∞ is given by taking the interval $T_0 = [0, 1]$, removing the open middle third (T_1), removing the middle third of each of the two remaining pieces (T_2), and continuing this procedure to infinitum.

Separated sets: two sets A, B in $\mathbb{X}$ are separated if both $A \cap \bar{B}$ and $B \cap \bar{A}$ are empty.

Connected sets: $\mathbb{E}$ is connected if $\mathbb{E}$ is not the union of two separated sets in $\mathbb{E}$.

$[a, b]$ is connected. (how to prove it?)

3 Numerical Sequences and Series

Sequence Convergence

Sequences $\{p_n\}$ converges if $\exists p \in X$ such that $\forall \epsilon > 0, \exists N$ such that $n \geq N \Rightarrow d(p_n, p) < \epsilon$. Write $p_n \to p$ or $\lim\limits_{n\to\infty} p_n = p$.

$p_n \to p$ and $p_n \to p' \Rightarrow p = p'$.

p_n converges $\Rightarrow \{p_n\}$ bounded.

Convergence properties: let sequences $\{s_n\}$, $\{t_n\} \in C$, $\{s_n\} \to s, \{t_n\} \to t$, then:

$\lim\limits_{n\to\infty} (s_n + t_n) = s + t$.

$\lim\limits_{n\to\infty} cs_n = cs$.

$\lim\limits_{n\to\infty} (c + s_n) = c + s$.

$\lim\limits_{n\to\infty} s_n t_n = st$.

Subsequence

$\{p_n\}$ is a sequence, let $n_1 < n_2 < \dots$ in N, then $\{p_{n_i}\}$ is a subsequence.

If $p_n \to p$, every subsequence converge to p.

Sequentially compact: a metric space is sequentially compact if every sequence has a convergent subsequence.

If $\mathbb{X}$ is compact, then $\mathbb{X}$ is sequentially compact.

Bolzano-Welerstrass Theorem Every bounded sequence in $\mathbb{R}^k$ contains a convergent subseqence.

Cauchy Sequence

A sequence $\{p_n\}$ is a Cauchy sequence if for every $\epsilon > 0, \exists$ an N such that for $\forall m, n \geq N \Rightarrow d(p_m, p_n) < \epsilon$.

If $\{p_n\}$ converges, then $\{p_n\}$ is Cauchy. (how to prove it?)

Complete Spaces

A metric spaces $\mathbb{X}$ is completed if every Cauchy sequence converges to point of $\mathbb{X}$.

Compact metric spaces are complete.(how to prove it?)

$\mathbb{R}^n$ is complete.

Every metric space $(\mathbb{X}, d)$ has a completion $(\mathbb{X}^*, d)$.

Bounded monotonic sequences converges.

Series

A series is an infinite ordered set of terms combined together by the addition operator.

Given series $\{a_n\}$, $s_n = \sum_{k=1}^{n} a_k$ is the nth partial sum.

$\sum_{n=1}^{\infty} a_n$ is an infinite series.

$\sum a_n$ converges if and only if $\forall \epsilon > 0, \exists N$ such that $m, n > N \Rightarrow |\sum_{k=n}^{m} a_n| < \epsilon$.

If $a_n \geq 0$, then $\sum a_n$ converges if and only if the partial sums are bounded.

Comparison Test:

(a) if $|a_n| \leq c_n$ for n large enough, and $\sum c_n$ converges, then $\sum a_n$ converges as well.

(b) if $a_n \geq d_n \geq 0$ for n large enough, then if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Geometric Series: $\sum_{n=1}^{\infty} x^n$

(a) if $|x| < 1$, then $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$.

(b) if $|x| \geq 1$, then the series diverges.

Suppose $a_1 > a_2 > a_3 > \cdots > 0$, then $\sum a_n$ converges is equivalent to $\sum s^k a_{2k}$ converges. (how to prove it?)

Series Convergence Tests

Given a series $\sum a_n$, let $\alpha = \lim sup(\sqrt[n]{|a_n|})$, then

if $\alpha < 1$, the series converges,

if $\alpha > 1$, the series diverges,

if $\alpha = 1$, the test cannot decide.

Ratio Test: $\sum a_n$ converges if $\lim sup \frac{a_{n+1}}{a_n} < 1$; diverges if $\frac{a_{n+1}}{a_n} > 1$ for n large enough.

Power Series: For c_n complex, then a power series is in the form of $\sum_{n=0}^{\infty} c_n z^n$.

Let $\alpha = \lim sup(\sqrt[n]{|a_n|})$ and $R = \frac{1}{\alpha}$ (convergene radius), then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| < R$ and diverges if $|Z| > R$.

Summation by parts: let $A_n = \sum_{k=0}^{n} a_n$ for $n > 0, A_t = 0$, then
$\sum_{p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$.

If $A_n = \sum_{k=0}^{n} a_n$ is bounded and b_n are decreasing and approaching 0, then $\sum a_n b_n$ converges.

Absolute convergence: a series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

If $\sum a_n = A$ and $\sum b_n = B$ converges absolutely, then the sum $\sum c_n$ converges to AB, in which $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

4 Continuity

Functin Limit

Let f be the function: $E \to Y$. If $\exists p \in Y$ such that for every $\epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in E, 0 < d(x, p) < \delta \to d(f(x), q) < \epsilon$.

$\lim_{x \to p} f(x) = q$ is true iff. for all sequences $\{p_n\} \in E$ such that $p_n \neq p$ but $p_n \to p$ we have $f(p_n) \to q$.

Algebraic Limit Theorem: Let $f : A \to R$ and $g : A \to R$ are continuous at $c \in A$, then $kf(x), f(x) + g(x), f(x)g(x), f(x)/g(x)$ is continous at c.

Continuous Function

If $p \in E \subset X$, function f is continuous at p if for every $\epsilon > 0, \exists \delta > 0$ such that $\forall x \in E, d(x, p) < \delta \to d(f(x), f(p)) < \epsilon$.

$f : X \to Y$ is continuous for all points

$\Leftrightarrow$ $\forall$ open sets $U \in Y$, its inverse image $f^{-1}(U)$ is open in X.

$\Leftrightarrow$ the inverse image of closed sets are closed.

If $f : X \to \mathbb{R}$ is continuous and $K \subseteq X$ is compact, then $f(K)$ is compact.

The Extreme Value Theorem if $f : X \to Y$ is bijection and continous, X is compact, then $f^{-1}(X)$ is continous.

Uniformly Continuos

if $\forall \epsilon > 0$, $\exists \delta > 0$, such that $\forall x$ and $p \in X$, $d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$.

A continuous function f is uniformly continuous on a compact set K.

For a continuous function $f : A \to \mathbb{R}$ if $E \in A$ is connected, then $f(E)$ is connected as well.

Intermediate Value Theorem

Discontinuous Function

Monotonic Function

5 Differentiation

Derivatives

Let $f : A \to \mathbb{R}$ be a function where A is an interval. For $c \in A$, the derivative of f at c is defined by $f'(c) = lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

If f is differentiable at $c \in A$ then f is continuous at c as well.

$(f + g)'(c) = f'(c) + g'(c)$

$(kf)'(c) = kf'(c)$ for constant $k \in \mathbb{R}$

$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

$(f/g)'(c) = \frac{f'(c)g(c)f(c)g'(c)}{[g(c)]^2}$

Interior Extremum Theorem

Mean Value Theorem

f $f : [a, b] \to R$ is continuous on $[a, b]$, differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$\frac{\infty}{\infty}$ Case